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A Note on Multiplicities of Singular Representations in the Discrete Spectrum of $Sl(2, R)/\Gamma$

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1. INTRODUCTION

Let G be a semi-simple Lie group with maximal compact subgroup K. It is well known that we have for a discrete torsion free cocompact subgroup Γ and an irreducible finite dimensional representation (ρ, V)

$$H^*(K\backslash G/\Gamma, \tilde{V}) = \bigoplus H^*(g, K, \Pi \otimes V)^{m(\Gamma, \Pi)},$$

where $\tilde{V}$ is the local system induced by V. We sum over all unitary representations Π with multiplicity $m(\Gamma, \Pi)$ in $L^2(G/\Gamma)$.

This formula can be used in two ways. First of all one can use representation theoretic methods to prove geometric vanishing theorems [3, 10], and second one can use geometric methods, for example, index theorems, to prove that certain representations appear in $L^2(G/\Gamma)$ with positive multiplicities [7, 8, 6]. Unfortunately the cohomology of many unitary representations with integral infinitesimal character vanishes and hence these representations cannot be detected with this method. In particular interesting unitary representations like many unipotent representations have trivial (g, K)-cohomology and hence are not related to the cohomology of a local system.

Let us look at an example. For $G = SL(2, R)$ the only unitary representations with nontrivial cohomology are discrete series representations, whereas the two representations Π_0^+, Π_0^- which are limits of discrete series representations have singular infinitesimal character and thus do not

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contribute to $H^*(K\backslash G/\Gamma, \tilde{V})$ for any finite dimensional representation V. Hence techniques using the cohomology of the space $K\backslash G/\Gamma$ give us no information regarding the occurrence of Π_0^+ and Π_0^- in $L^2(G/\Gamma)$.

For some groups G and lattices Γ it has been shown using the oscillator representation that certain representations with singular integral infinitesimal character do occur in the spectrum of $L^2(G/\Gamma)$ [3, pp. 228–257]. However, nothing is known about their multiplicities nor about a geometric interpretation of their occurrence.

In this article we attack this problem for a cocompact subgroup Γ of $G = SL(2, R)$. Using the Casimir we construct a differential operator $\mathscr{C}_l$ on the sections of a 2-dimensional vector bundle over $K\backslash G/\Gamma$. Let S be the sheaf of sections annihilated by $\mathscr{C}_l$. Then (see 5.1)

$$\dim H^*(K\backslash G/\Gamma, S) = 2(m(\Gamma, \Pi_0^+) + m(\Gamma, \Pi_0^-)).$$

Our construction does not make use of the complex structure of $K\backslash G/\Gamma$, but uses only the K-types and the infinitesimal character of Π_0^+ and Π_0^+. So it can be extended to other compact locally symmetric spaces and other families of singular unipotent representations [1].

The paper is organized as follows: In Section 2 we collect some information about tensor products of representations of $Sl(2, R)$ with finite dimensional representations. In Section 3 we construct an elliptic differential operator acting on the sheaf of sections of a bundle. In Section 4 we compute the cohomology of the sheaf of solutions of this differential operator. We finish the proof of Theorem 1.1 in Section 4.

Throughout this article we will use the following notation and assumptions:

G the special linear group in dimension 2,

K the special orthogonal group in dimension 2,

Γ a discrete cocompact subgroup of G,

V a finite dimensional vectorspace,

$\rho: G \to \mathrm{End}(V)$ a finite dimensional irreducible representation of G.

2. Tensor Products and Cohomology

In this section we collect all the necessary results about tensor products of unitary representations of G with a finite dimensional representation ρ and compute the cohomology of all their composition factors. Since most of the results are well known we omit many proofs and refer to [9] for details.

The irreducible unitary representations of G with integral infinitesimal character can be described as follows:

1. *Discrete series representations* Π_n^+, Π_n^-. Here $n > 0$ refers to the dimension of the finite dimensional representation with the same infinitesimal character and the sign denotes the holomorphic, respectively antiholomorphic representation.

2. Two representations Π_0^+, Π_0^- which are *limits of discrete series representations*. Again the sign denotes the holomorphic, respectively antiholomorphic representation.

3. *A unitarily induced principal series representation* U_0 which has a trivial K-type.

4. The *trivial representation.*

The set of irreducible unitary representations with integral infinitesimal character will be denoted by G_I.

LEMMA 2.1. *Let Π be an irreducible unitary representation and V a finite dimensional irreducible representation of G. The following are equivalent:*

- *$\Pi \otimes V$ has a composition factor with integral infinitesimal character.*
- *Π has integral infinitesimal character.*
- *Every composition factor of $\Pi \otimes V$ has integral infinitesimal character.*

Let V_2 denote the 2-dimensional irreducible representation of G. To simplify the notation we will write Π_n^σ instead of Π_n^+, respectively Π_n^-.

LEMMA 2.2. *The tensor product of a representation in G_I with V_2 has the following structure.*

- *The tensor product $U_0 \otimes V_2$ is indecomposable. Its two composition factors are both isomorphic to an irreducible nonunitary principal series representation.*
- *The tensor product $\Pi_0^\sigma \otimes V_2$ is indecomposable. Its three composition factors are isomorphic to Π_1^σ and the trivial representation. The socle filtration has length 3 with socle Π_1^σ and lid Π_1^σ.*
- *The tensor product $\Pi_n^\sigma \otimes V_2, n > 0$, is a direct sum of Π_{n-1}^σ and Π_{n+1}^σ.*

LEMMA 2.3. *Let C be the Casimir element in $U(g)$. There exists an exact sequence*

$$0 \to K^\sigma \to \Pi_0^\sigma \otimes V_2 \to \Pi_1^\sigma \to 0,$$

where K^σ is the kernel of C.

Proof. Since C is an endomorphism of $\Pi_0^\sigma \otimes V_2$ its image is a submodule of $\Pi_0^\sigma \otimes V_2$. So by 2.2 it is equal to Π_1^σ. Q.E.D.

Remark. Although we have by Wigner's lemma [3] that

$$H^*(g, K, \Pi_0^\sigma \otimes V_2) = 0$$

we can prove using the short exact sequence in Lemma 2.3 that

$$\dim H^i(g, K, K^\sigma) = 1$$

iff $i = 2$. Otherwise it has dimension 0.

3. An Elliptic Differential Operator

In this section we construct an elliptic second order differential operator on a bundle with base $K\backslash G/\Gamma$ and fiber V_2. Its definition is inspired by the action of the Casimir on the tensor product of two representations and it is equal up to terms of lower order to the Laplace operator.

The torsion free discrete subgroup Γ is the fundamental group of the locally symmetric space $K\backslash G/\Gamma$. Consider the G-module V_2 as representation of Γ and let $\mathscr{V} = K\backslash G \times_\Gamma V_2$ be a vector bundle with base $K\backslash G/\Gamma$ and fiber V_2. We will define an action of the Casimir $C \in U(g)$ on the sheaf $\mathscr{C}^\infty(\mathscr{V})$ of sections of $\mathscr{V}$.

We start by recalling sme facts about the vector bundle $\mathscr{V}$ and its global sections [3].

Consider V_2 as a Γ-module. Then $\mathscr{W} = G \times_\Gamma V_2$ is a homogeneous vector bundle on G/Γ with fiber V_2 and structure group Γ. Since V_2 is a G-module, the bundle $\mathscr{W}$ is a G-space and hence the C^∞-sections of $\mathscr{W}$ are isomorphic as $U(g)$-module to the tensor product $C^\infty(G/\Gamma) \otimes V_2$ via the map

$$\Psi: (f, v) \mapsto f_v,$$

where $g \in G$, $f \in C^\infty(G/\Gamma)$, $v \in V_2$, and

$$f_v(g) = f(g) \cdot g^{-1} v.$$

Let $C^\infty(\mathscr{W})^K$ denote the K-invariant sections of $\mathscr{W}$. We have an isomorphism $\Lambda: C^\infty(\mathscr{W})^K \to \mathscr{C}^\infty(\mathscr{V})$ and hence

$$\Lambda\Psi: (C^\infty(G/\Gamma) \otimes V_2)^K \to C^\infty(\mathscr{V})$$

is an isomorphism.

The Casimir $C \in U(g)$ defines a $U(g)$-module map

$$C_l: (C^\infty(G/\Gamma) \otimes V_2) \to (C^\infty(G/\Gamma) \otimes V_2).$$

Since it is Ad(K)-invariant it defines a map

$$C_l: (C^\infty(G/\Gamma) \otimes V_2)^K \to (C^\infty(G/\Gamma) \otimes V_2)^K.$$

Hence $C_l = \Delta\Psi C_l(\Lambda\Psi)^{-1}$ is an endomorphism of $\mathscr{C}^\infty(\mathscr{V})$.

We can define another action of the Casimir C on $\mathscr{C}^\infty(\mathscr{V})$. Since C is Ad(Γ)-invariant, the action of C from the right defines a $U(g)$-module map

$$C_r: C^\infty(G/\Gamma) \otimes V_2 \to C^\infty(G/\Gamma) \otimes V_2$$

which acts identically on the second factor and so it defines a linear operator $\mathscr{C}_r = \Lambda\Psi C_r(\Lambda\Psi)^{-1}$ on $\mathscr{C}^\infty(\mathscr{V})$.

LEMMA 3.1. *$\mathscr{C}_r$ and $\mathscr{C}_l$ are differential operators.*

Proof. To prove this we use Peetre's characterisation of a kth order differential operator on a bundle [5].

We write $P: G/\Gamma \to K\backslash G/\Gamma$ for the natural projection. If $f \in C_c^\infty(\mathscr{V})$ we write F for a section in $C^\infty(\mathscr{W})$ defined by $F(g) = f(P(g))$.

Since $\Psi C_l \Psi^{-1}$ and $\Psi C_r \Psi^{-1}$ are differential operators and since they commute with K we have

$$\text{supp}(\mathscr{C}_l f) = P(\text{supp}(\Psi C_l \Psi^{-1} F)) \subset P(\text{supp}(F)) = \text{supp}(f). \quad \text{Q.E.D.}$$

PROPOSITION 3.2. *$\mathscr{C}_r$ is an elliptic operator.*

Proof. We may identify C_r and the operator

$$C \otimes 1: (C^\infty(G/\Gamma) \otimes V_2)^K \to (C^\infty(G/\Gamma) \otimes V_2)^K,$$

where

$$(C \otimes 1)(f \otimes v) = C(f) \otimes v.$$

So

$$\begin{aligned}(\Psi\Lambda(C_r(f \otimes v)))(g) &= \Psi(Cf(g) \otimes g^{-1}v) \\ &= Cf(Kg) \otimes (Kg)^{-1}v \\ &= \mathscr{C}_r(\Psi\Lambda(f(g) \otimes v)).\end{aligned}$$

Since the Casimir C act on the functions on the upper half plane as the Laplace operator, the symbol of $\mathscr{C}_r$ is a product of the symbol of the Laplace operator and the identity matrix on the fiber. The Laplace operator is elliptic and so $\mathscr{C}_r$ is elliptic. Q.E.D.

PROPOSITION 3.3. *$\mathscr{C}_l$ is an elliptic second order differential operator.*

Proof. Let C be the Casimir element in the universal enveloping algebra $U(g)$. We pick a basis X, Y, Z of the Lie algebra $sl(2, R)$ so that $C = X^2 + ZY + YZ$. In this case X is a generator of the Lie algebra k of K. Analysing the tensor product action of C we see that it equals

$$C \otimes 1 + 1 \otimes C + X \otimes X + Y \otimes Z + Z \otimes Y.$$

The part contributing to the highest order term of $\mathscr{C}_l$ is $C \otimes 1$, which is equal to C_r. Hence by 3.2, $\mathscr{C}_l$ is elliptic. Q.E.D.

Let $\Gamma(S)$ be the kernel of $\mathscr{C}_l\colon C^\infty(\mathscr{V}) \to C^\infty(\mathscr{V})$. We refer to S as the *sheaf of solutions of the operator $\mathscr{C}_l$.*

4. A Vanishing Theorem

In this section we study the sheaf cohomology of S.

THEOREM 4.1. *Let S be the sheaf of solutions of the operator $\mathscr{C}_l$. Then*

$$\dim H^0(K\backslash G/\Gamma, S) = \dim H^1(K\backslash G/\Gamma, S)$$

and $H^i(K\backslash G/\Gamma, S) = 0$ otherwise.

Proof. Since the inhomogeneous equation

$$\mathscr{C}_l f = g$$

is locally solvable [4], we have the exact sequence

$$0 \longrightarrow S \longrightarrow \mathscr{C}^\infty(\mathscr{V}) \xrightarrow{\mathscr{C}_l} \mathscr{C}^\infty(\mathscr{V}) \longrightarrow 0.$$

Hence

$$0 \longrightarrow \Gamma(S) \longrightarrow C^\infty(\mathscr{V}) \xrightarrow{\mathscr{C}^l} C^\infty(\mathscr{V}) \longrightarrow H^1(X, S) \longrightarrow 0.$$

Since $\mathscr{C}_l$ is elliptic and $M = K\backslash G/\Gamma$ is compact we have $\dim H^0(X, S) = \dim \Gamma(S) = \dim \ker(\mathscr{C}^l)$ and $\dim H^1(M, S) = \dim \operatorname{coker}(\mathscr{C}^l)$. Furthermore $\dim H^i(M, S) = 0$ for $i > 1$. The symbols of $\mathscr{C}_l$ and $\mathscr{C}_r$ are equal and thus their index. But the index of $\mathscr{C}_r$ is zero since $L^2(G/\Gamma)$ decomposes into a direct sum of irreducible unitary representations. Q.E.D.

Remark. The locally symmetric space $M = K\backslash G/\Gamma$ is an analytic compact manifold and the operator $\mathscr{C}_l$ defines on every analytic coordinate patch an elliptic system of equations with analytic coefficients. Since all solutions of elliptic systems of equations with analytic coefficients are analytic we can consider S as a subsheaf of the sheaf $\mathscr{A}_M$ of analytic functions on M.

We denote the sheaf of differential operators on M with analytic coefficients by $\mathscr{D}_M$. Let $\mathscr{L}_M$ be the $\mathscr{D}_M$-module defined by $\mathscr{C}_l$. This module is coherent [2] and for $i > 1$

$$\operatorname{Ext}^i_{\mathscr{D}_M}(\mathscr{L}_M, \mathscr{A}_M) = 0.$$

Hence

$$H^i(M, S) = \operatorname{Ext}^i_{\mathscr{D}_M}(\mathscr{L}_M, \mathscr{A}_M).$$

5. Proof of the Main Theorem

In this section we will complete the proof of the

THEOREM 5.1. *Let Γ be a discrete torsion free cocompact subgroup of G and let S be the sheaf of solutions of the partial differential operator* $C_l: C^\infty(\mathscr{V}) \to C^\infty(\mathscr{V})$. *Then*

$$\dim H^*(K\backslash G/\Gamma, S) = 2(m(\Gamma, \Pi_0^+) + m(\Gamma, \Pi_0^-)).$$

We first note that $H^0(K\backslash G/\Gamma, S) = \Gamma(S)$, i.e., it is the space of global solutions of the equation $\mathscr{C}_l f = 0$.

Suppose that Π_0^σ is a subrepresentation of $L^2(G/\Gamma)$. Then the trivial K-type occurs with multiplicity 1 in $\Pi_0^\sigma \otimes V_2$ and by 2.3 it is in the kernel of the operator $\mathscr{C}_l$: $C^\infty(G/\Gamma) \otimes V_2 \to C^\infty(G/\Gamma) \otimes V_2$. Hence

$$\begin{aligned}\dim \operatorname{Kern}(\mathscr{C}_l) &\geqslant \dim \operatorname{Hom}_{U(g)}(\Pi_0^+, L^2(G/\Gamma)) \\ &\quad + \dim \operatorname{Hom}_{U(g)}(\Pi_0^-, L^2(G/\Gamma)) \\ &= m(\Gamma, \Pi_0^+) + m(\Gamma, \Pi_0^-).\end{aligned}$$

Every solution of $\mathscr{C}_l f = 0$ is K-invariant and finite under the center of $U(g)$, so it is contained in a finite sum of subrepresentations of $L^2(G/\Gamma) \otimes V_2$. Since we showed in Section 2 that the representations Π_0^σ are the only unitary representations in the kernel of $\mathscr{C}_l$ which have a trivial K-type we have

$$\begin{aligned}\dim \operatorname{Kern}(\mathscr{C}_l) &= \dim \operatorname{Hom}_{U(g)}(\Pi_0^+, L^2(G/\Gamma)) \\ &\quad + \dim \operatorname{Hom}_{U(g)}(\Pi_0^-, L^2(G/\Gamma)).\end{aligned}$$

So by 4.1 the theorem follows. Q.E.D.

Remark 1. In the proof we did not use the hermitian structure of G/Γ. So most of the techniques of the proof can be extended to more general compact locally symmetric spaces. In this case the Casimir has to be replaced by the action of the center. The representations contributing to the cohomology of the sheaf of solutions are certain unipotent representations in the sense of Arthur [1].

Remark 2. It would also be interesting to do a similar analysis for other K-types or for multiple tensor products in a situation which simulates multiple wall crossings.

Remark 3. An interesting problem is to extend the above techniques to the noncompact situation. In that case we have to analyse the sheaf of L^2-solutions. Probably it is possible to set up a similar theory in this case.

Remark 4. Using the standard machinary of D-modules we can also construct a complex of sheaves of differential forms with coefficients in a sheaf which has the same cohomology as S, [2].

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